# HOSSAM GHANEM

## (2) 2.3 Techniques for finding limits (A)

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| $(x+a)^2 = x^2 + 2ax + a^2$ | $x^2 - a^2 = (x-a)(x+a)$<br>$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$<br>$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$ |

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| $\sqrt{x^2} = \lvert x \rvert$ | | |
| $\lvert x \rvert = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases}$ | $\lvert x-3 \rvert = \begin{cases} -(x-3) & \text{if} \quad x < 3 \\ 0 & \text{if} \quad x = 3 \\ x-3 & \text{if} \quad x > 3 \end{cases}$ | $\lvert x+5 \rvert = \begin{cases} -(x+5) & \text{if} \quad x < -5 \\ 0 & \text{if} \quad x = -5 \\ x+5 & \text{if} \quad x > -5 \end{cases}$ |

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| $\frac{\lvert x-a \rvert}{x-a} = \begin{cases} -1 & \text{if} \quad x < a \\ 1 & \text{if} \quad x > a \end{cases}$ | | |
| $\frac{\lvert x \rvert}{x} = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$ | $\frac{\lvert x-2 \rvert}{x-2} = \begin{cases} -1 & \text{if} \quad x < 2 \\ 1 & \text{if} \quad x > 2 \end{cases}$ | $\frac{\lvert x+4 \rvert}{x+4} = \begin{cases} -1 & \text{if} \quad x < -4 \\ 1 & \text{if} \quad x > -4 \end{cases}$ |

| Example 1<br>60 October 31 , 2011 | ( 2 points) Find the limit , if it exists . $\lim_{x\to 2}\frac{\sqrt{2x+5}-2}{x^2-9}$ |
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## Solution

$$L=\lim_{x\to 2}\frac{\sqrt{2x+5}-2}{x^2-9}=\frac{\sqrt{2(2)+5}-2}{(2)^2-9}=\frac{\sqrt{9}-2}{4-9}=\frac{3-2}{4-9}=\frac{1}{-5}=-\frac{1}{5}$$

| Example 2<br>48 March 25, 2008 A | Find the limit , if it exists $\lim_{x\to 1}\frac{x^3-1}{x^2-3x+2}$ |
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## Solution

$$\lim_{x\to 1}\frac{x^3-1}{x^2-3x+2}=\frac{1-1}{1-3+2}=\frac{0}{0}$$

$$\lim_{x\to 1}\frac{x^3-1}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(x-2)}=\lim_{x\to 1}\frac{(x^2+x+1)}{(x-2)}=\frac{1+1+1}{1-2}=-3$$

| Example 3<br>46 Date: July 5, 2007 | Find the following limit , if it exists $\lim_{x\to 3}\left(\frac{x^3-27}{\sqrt{x}-\sqrt{3}}\right)$ |
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## Solution

$$\lim_{x\to 3}\left(\frac{x^3-27}{\sqrt{x}-\sqrt{3}}\right)=\frac{3^3-27}{\sqrt{3}-\sqrt{3}}=\frac{27-27}{\sqrt{3}-\sqrt{3}}=\frac{0}{0}$$

$$\lim_{x\to 3}\left(\frac{x^3-27}{\sqrt{x}-\sqrt{3}}\right)=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{\sqrt{x}-\sqrt{3}}=\lim_{x\to 3}\frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})(x^2+3x+9)}{(\sqrt{x}-\sqrt{3})}$$

$$=\lim_{x\to 3}(\sqrt{x}+\sqrt{3})(x^2+3x+9)=(\sqrt{3}+\sqrt{3})(9+9+9)=2\sqrt{3}\cdot 27=54\sqrt{3}$$

| Example 4<br>23 May 26. 2002 | Evaluate the following limit $\lim_{x\to 27}\frac{3x^{\frac{2}{3}}-27}{x^{\frac{1}{3}}-3}$ |
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## Solution

$$\lim_{x\to 27}\frac{3x^{\frac{2}{3}}-27}{x^{\frac{1}{3}}-3}=\lim_{x\to 27}\frac{3(27)^{\frac{2}{3}}-27}{(27)^{\frac{1}{3}}-3}=\lim_{x\to 27}\frac{3(9)-27}{3-3}=\frac{0}{0}$$

$$\therefore \lim_{x\to 27}\frac{3x^{\frac{2}{3}}-27}{x^{\frac{1}{3}}-3}=\lim_{x\to 27}\frac{3\left[\left(x^{\frac{1}{3}}\right)^2-9\right]}{x^{\frac{1}{3}}-3}=\lim_{x\to 27}\frac{3\left(x^{\frac{1}{3}}-3\right)\left(x^{\frac{1}{3}}+3\right)}{\left(x^{\frac{1}{3}}-3\right)}=\lim_{x\to 27}3\left(x^{\frac{1}{3}}+3\right)=3(3+3)=18$$

$$(27)^{\frac{1}{3}}=\sqrt[3]{27}=3$$

| Example 5<br>33 October 25,2001<br>A | Find the limit , if it exists $\lim_{x\to 1}\frac{x-1}{\sqrt[3]{x}-1}$ |
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## Solution

$$\lim_{x\to 1}\frac{x-1}{\sqrt[3]{x}-1}=\frac{1-1}{1-1}=\frac{0}{0}$$

$$\lim_{x\to 1}\frac{x-1}{\sqrt[3]{x}-1}=\lim_{x\to 1}\frac{\left[\left(x^{\frac{1}{3}}\right)^3-1\right]}{x^{\frac{1}{3}}-1}=\lim_{x\to 1}\frac{\left(x^{\frac{1}{3}}-1\right)\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1\right)}{x^{\frac{1}{3}}-1}=\lim_{x\to 1}\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1\right)=1+1+1=3$$

| Example 6<br>29 Feb. 24, 2000 | Evaluate the following limit $\lim_{x\to 2}\frac{1-\sqrt{\frac{x}{2}}}{1-\frac{x}{2}}$ |
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## Solution

$$\lim_{x\to 2}\frac{1-\sqrt{\frac{x}{2}}}{1-\frac{x}{2}}=\frac{1-1}{1-1}=\frac{0}{0}$$

$$\therefore\ \lim_{x\to 2}\frac{1-\sqrt{\frac{x}{2}}}{1-\frac{x}{2}}=\lim_{x\to 2}\frac{\left(1-\sqrt{\frac{x}{2}}\right)}{\left(1-\sqrt{\frac{x}{2}}\right)\left(1+\sqrt{\frac{x}{2}}\right)}=\lim_{x\to 2}\frac{1}{\left(1+\sqrt{\frac{x}{2}}\right)}=\frac{1}{1+1}=\frac{1}{2}$$

| Example 7<br>12 November 2,<br>1995 | Find the following limit , if it exists $\lim_{x\to 1}\frac{x^2+2x-3}{x\lvert x-1\rvert}$ |
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## Solution

$$L=\lim_{x\to 1}\frac{x^2+2x-3}{x\lvert x-1\rvert}=\frac{1+2-3}{(1)\lvert 1-1\rvert}=\frac{0}{0}$$

$$\lim_{x\to 1}\frac{x^2+2x-3}{x\lvert x-1\rvert}=\lim_{x\to 1}\frac{(x+3)(x-1)}{x\lvert x-1\rvert}$$

$$L_1=\lim_{x\to 1^+}\frac{x^2+2x-3}{x\lvert x-1\rvert}=\lim_{x\to 1^+}\frac{(x+3)(x-1)}{x(x-1)}=\lim_{x\to 1^+}\frac{(x+3)}{x}=\frac{4}{1}=4 \quad \to (1)$$

$$L_2=\lim_{x\to 1^-}\frac{x^2+2x-3}{x\lvert x-1\rvert}=\lim_{x\to 1^-}\frac{(x+3)(x-1)}{-x(x-1)}=\lim_{x\to 1^-}\frac{(x+3)}{-x}=-\frac{4}{1}=-4 \quad \to (2)$$

from (1), (2)

$\therefore\ L_1 \neq L_2$

$\therefore\ L=\lim_{x\to 1}\frac{x^2+2x-3}{x\lvert x-1\rvert}$ D. N. E

| Example 8<br>24 November 3, 1998 | Evaluate the following limit $\lim_{x\to 2}(x-2)\sqrt{1+\frac{1}{(x-2)^2}}$ |
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## Solution

$$L=\lim_{x\to 2}f(x)=\lim_{x\to 2}(x-2)\sqrt{1+\frac{1}{(x-2)^2}}=0\cdot\infty$$

$$L=\lim_{x\to 2}f(x)=\lim_{x\to 2}(x-2)\sqrt{\frac{(x-2)^2+1}{(x-2)^2}}=\lim_{x\to 2}\frac{x-2}{|x-2|}\sqrt{(x-2)^2+1}$$

$$L_1=\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}\frac{x-2}{x-2}\sqrt{(x-2)^2+1}=\lim_{x\to 2^+}\sqrt{(x-2)^2+1}=1 \quad \rightarrow (1)$$

$$L_2=\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}\frac{x-2}{-(x-2)}\sqrt{(x-2)^2+1}=\lim_{x\to 2^-}-\sqrt{(x-2)^2+1}=-1 \quad \rightarrow (2)$$

from (1), (2)

$\therefore \lim_{x\to 2^+}f(x)\neq\lim_{x\to 2^-}f(x)$

$$\therefore \lim_{x\to 2}(x-2)\sqrt{1+\frac{1}{(x-2)^2}} \quad D.N.E$$

| Example 9<br>38 January 15, 2011 | Suppose that $\lim_{x\to a}f(x)$ and $\lim_{x\to a}g(x)$ both exist.<br>if $\lim_{x\to a}[f(x)+g(x)]=5$ and $\lim_{x\to a}[f(x)-g(x)]=1$<br>then find $\lim_{x\to a}[f(x)g(x)]$ |
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## Solution

Let $\lim_{x\to a}f(x)=\mathrm{L}$ and $\lim_{x\to a}g(x)=\mathrm{K}$

$\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=\mathrm{L}+\mathrm{K}$

$\therefore \mathrm{L}+\mathrm{K}=5 \quad \rightarrow \boxed{1}$

$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=\mathrm{L}-\mathrm{K}$

$\therefore \mathrm{L}-\mathrm{K}=1 \quad \rightarrow \boxed{2}$

$\boxed{1}+\boxed{2}$ $\quad 2\mathrm{L}=6 \quad \rightarrow \quad L=3$

$\boxed{1}-\boxed{2}$ $\quad 2\mathrm{K}=4 \quad \rightarrow \quad K=2$

$$\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=\mathrm{L}\cdot\mathrm{K}=3\cdot 2=6$$

# Homework

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| 1 | Find the limit , if it exists | $\lim_{x \to -2} \frac{x^3+8}{x+2}$ | 30 October 19, 2000 A |
| 2 | Evaluate the following limit(if it exists) | $\lim_{x \to 1} \frac{\sqrt[3]{x}-1}{x-1}$ | 6 January 6, 1993 |
| 3 | Evaluate the following limit (if it exists) | $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ | 32 March 22, 2001 |
| 4 | Find | $\lim_{x \to 1} \frac{\sqrt{x}-1}{x^2-1}$ | 36 April 19,2003 A |
| 5 | Find the limit , if it exists | $\lim_{x \to 0} \frac{3\|x\|}{x+x^3}$ | 3 March 19, 1992 |
| 6 | Evaluate the following limit(if it exists) | $\lim_{x \to 1} \frac{3\|x-1\|}{x-x^3}$ | 8 October 28, 1993 |
| 7 | Find the limit , if it exists | $\lim_{x \to 0} x\sqrt{1+\frac{1}{x^2}}$ | 3 December 30, 1991 |
| 8 | Evaluate the following limit(if it exists) | $\lim_{x \to 3} (x-3)\sqrt{\frac{x^2+1}{(x-3)^2}}$ | |
| 9 | Evaluate the following limit(if it exists) | $\lim_{x \to 8} \frac{3x^{\frac{2}{3}}-12}{x^{\frac{1}{3}}-2}$ | 12 January 19, 1995 |
| 10 | Evaluate the following limit | $\lim_{x \to 2} (x-2)\sqrt{\frac{x^2+1}{(x-2)^2}}$ | 1 November 1987 |
| 11 | Find the limit , if it exists | $\lim_{x \to 1} \frac{5\|x-1\|}{x^2+x-2}$ | 11 March 31, 1994 |

# Homework

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| 12 | Evaluate the following limit | $\lim_{x \to 0} x\sqrt{\frac{1}{x^2} - 1}$ | 15 July 15, 1996 |
| 13 | Find the following limit , if it exists | $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x^3 - 1}$ | |
| 14 | Find the limit , if it exists | $\lim_{x \to 0} x^2\sqrt{1 + \frac{1}{x^2}}$ | 14 March 28, 1996 |

# Homework

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| 10 | Find the following limit , if it exists $\lim_{x\to 2}(x-2)\sqrt{\frac{x^2+1}{(x-2)^2}}$ |

## Solution

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \quad (x-2)\sqrt{\frac{x^2+1}{(x-2)^2}}$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \quad \frac{x-2}{|x-2|}\sqrt{x^2+1}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-2}{x-2}\sqrt{x^2+1} = \lim_{x\to 2^+} \sqrt{x^2+1} = \sqrt{5} \quad \rightarrow (1)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-2}{-(x-2)}\sqrt{x^2+1} = \lim_{x\to 2^-} -\sqrt{x^2+1} = -\sqrt{5} \quad \rightarrow (2)$$

from (1), (2)

$$\therefore \lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$$

$$\therefore \lim_{x\to 2} \quad (x-2)\sqrt{\frac{x^2+1}{(x-2)^2}} \qquad D.N.E$$

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| 11 | Find the limit , if it exists $\lim_{x\to 1}\frac{5\|x-1\|}{x^2+x-2}$ |

## Solution

$$\lim_{x\to 1} \frac{5|x-1|}{x^2+x-2} = \lim_{x\to 1} \frac{5|x-1|}{(x+2)(x-1)} = \frac{0}{0}$$

$$\lim_{x\to 1^+} \frac{5|x-1|}{x^2+x-2} = \lim_{x\to 1^+} \frac{5(x-1)}{(x+2)(x-1)} = \lim_{x\to 1^+} \frac{5}{(x+2)} = \frac{5}{3} \quad \rightarrow (1)$$

$$\lim_{x\to 1^-} \frac{5|x-1|}{x^2+x-2} = \lim_{x\to 1^-} \frac{-5(x-1)}{(x+2)(x-1)} = \lim_{x\to 1^-} \frac{-5}{(x+2)} = -\frac{5}{3} \quad \rightarrow (2)$$

from (1), (2)

$$\therefore \lim_{x\to 1^+} \frac{5|x-1|}{x^2+x-2} \neq \lim_{x\to 1^-} \frac{5|x-1|}{x^2+x-2}$$

$$\therefore \lim_{x\to 1} \frac{5|x-1|}{x^2+x-2} \qquad D.N.E$$

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| 12 | Evaluate the following limit $\lim_{x\to 0} x\sqrt{\frac{1}{x^2}-1}$ |

## Solution

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sqrt{\frac{1}{x^2}-1} = \lim_{x\to 0} x\sqrt{\frac{1-x^2}{x^2}} = \lim_{x\to 0} \frac{x}{|x|}\sqrt{1-x^2}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x}\sqrt{1-x^2} = \lim_{x\to 0^+} \sqrt{1-x^2} = 1 \quad \to (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{-x}\sqrt{1-x^2} = \lim_{x\to 0^-} -\sqrt{1-x^2} = -1 \quad \to (2)$$

from (1), (2)

$\therefore \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

$\therefore \lim_{x\to 0} x\sqrt{\frac{1}{x^2}-1} \quad D.N.E$

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| 13 | Find the following limit , if it exists $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^3-1}$ |

## Solution

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^3-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^3-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)(x^2+x+1)}$$

$$= \lim_{x\to 1} \frac{1}{(\sqrt{x}+1)(x^2+x+1)} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

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| 14 | Find the limit , if it exists $\lim_{x\to 0} x^2\sqrt{1+\frac{1}{x^2}}$ |

## Solution

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\sqrt{1+\frac{1}{x^2}} = 0\cdot\infty$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\sqrt{\frac{x^2+1}{x^2}} = \lim_{x\to 0} \frac{x^2}{|x|}\sqrt{x^2+1}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2}{|x|}\sqrt{x^2+1} = \lim_{x\to 0^+} \frac{x^2}{x}\sqrt{x^2+1} = \lim_{x\to 0^+} x\sqrt{x^2+1} = 0 \quad \to (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2}{|x|}\sqrt{x^2+1} = \lim_{x\to 0^-} \frac{x^2}{-x}\sqrt{x^2+1} = \lim_{x\to 0^+} -x\sqrt{x^2+1} = 0 \quad \to (2)$$

from (1), (2)

$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0 \qquad \therefore \lim_{x\to 0} x\sqrt{\frac{1}{x^2}-1} = 0$